Convex Optimization in Machine Learning and Inverse Problems

Part 1: Applications of Sparse Optimization

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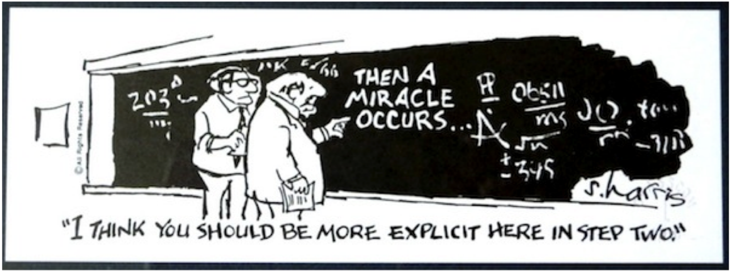

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Many **inference** problems are formulated as **optimization** problems:

- image reconstruction
- image restoration/denoising
- supervised learning
- unsupervised learning
- statistical inference
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Standard formulation:

- observed data: y
- unknown mathematical object (signal, image, vector, matrix,...): x
- inference criterion:

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Question 1: how to build g? Where does it come from?

Answer: from the application domain (machine learning, signal processing, inverse problems, system identification, statistics, computer vision, bioinformatics,...);
... examples ahead.

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Question 2: how to solve the optimization problem?

Answer: the focus of this tutorial.

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- $\tau \geq 0$: the **regularization parameter** (or constant).
- Since y is fixed, we often write simply $f(x) = h(x, y)$,

$$\min_{x} f(x) + \tau\psi(x)$$

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- Log-posterior: $\log\, p(x|y) = K(y) - h(x, y) - \tau\psi(x) = K(y) - g(x, y)$
- $\widehat{x}$ is a maximum a posteriori (MAP) estimate.

Regularizers

Inference criterion: $\min_x \; f(x) + \tau\psi(x)$

Typically, the unknown is a vector $x \in \mathbb{R}^n$
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Common regularizers impose/encourage one (or a combination of) the following characteristics:

- small norm (vector or matrix)
- sparsity (few nonzeros)
- specific nonzero patterns (*e.g.*, group/tree structure)
- low-rank (matrix)
- smoothness or piece-wise smoothness
- ...

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- Ivanov regularization: $$\begin{array}{ll} \min\limits_x & f(x) \\ \text{subject to} & \psi(x) \leq \delta \end{array}$$

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Under mild conditions, these are all *"equivalent"*.

Morozov and Ivanov can be written as Tikhonov using indicator functions (more later).

Which one is more convenient is problem-dependent.

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- Non-trivial cases: resort to optimization and regularization.
- Quadratic (Euclidean) losses and regularizers have a long and rich history: Gauss, Legendre, Wiener, Moore-Penrose, Tikhonov, ...

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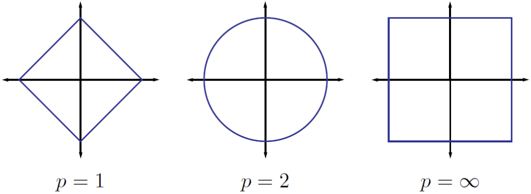
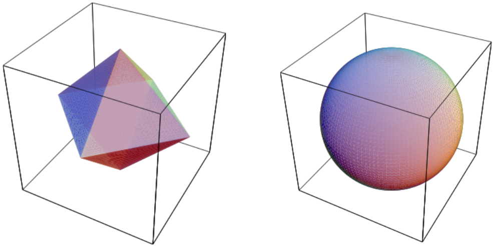
Also important (but not a norm): $\|x\|_0 = \lim_{p\to 0} \|x\|_p^p = |\{i : x_i \neq 0\}|$

Norm balls

Radius r ball in ℓ_p norm: $B_p(r) = \{x \in \mathbb{R}^n : \|x\|_p \leq r\}$





$p = 1$ $p = 2$

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- Can we hope to recover x? Yes! ...if x is sparse enough ($\|x\|_0 < k$) and A satisfies some **conditions**, using

$$\begin{array}{rl} \widehat{x} = & \arg\min_x \|x\|_0 \\ & \text{s.t. } Ax = y \end{array}$$

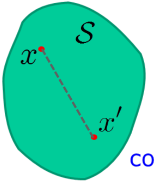
Several proofs, under different conditions (more later).

But, this is a hard problem! ℓ_0 “norm” is not **convex**.

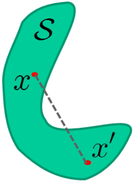
Review of Basics: Convex Sets

Convex and strictly convex sets

$\mathcal{S}$ is convex if $x, x' \in \mathcal{S} \;\Rightarrow \forall \lambda \in [0,1], \;\; \lambda x + (1-\lambda) x' \in \mathcal{S}$

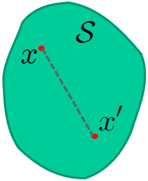


convex

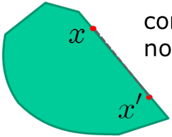


non-convex

$\mathcal{S}$ is strictly convex if $x, x' \in \mathcal{S} \;\Rightarrow \forall \lambda \in (0,1), \;\; \lambda x + (1-\lambda) x' \in \operatorname{int}(\mathcal{S})$



strictly convex



convex, but not strictly

Review of Basics: Convex Functions

Extended real valued function: $f : \mathbb{R}^N \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$

Domain: $\mathrm{dom}(f) = \{x : f(x) \neq +\infty\}$

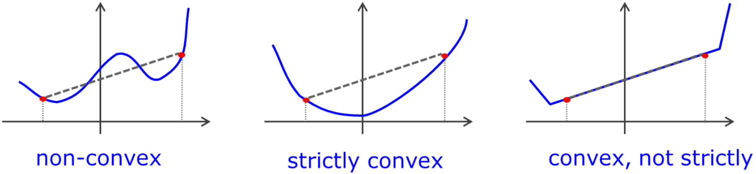
f is proper if $\mathrm{dom}(f) \neq \emptyset$

f is convex if

$$\forall \lambda \in [0,1], x, x' \in \mathrm{dom}(f) \;\; f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x')$$

f is strictly convex if

$$\forall \lambda \in (0,1), x, x' \in \mathrm{dom}(f) \;\; f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x')$$


non-convex

strictly convex

convex, not strictly

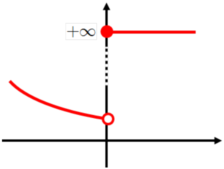
Lower Semi-Continuity: Why Is It Important?

A function $f : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ is lower semi-continuous (l.s.c.) if

$$\liminf_{x \rightarrow x_0} \geq f(x_0), \quad \text{for any } x_0 \in \text{dom}(f)$$

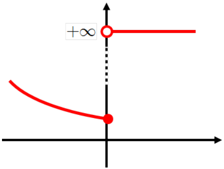
or, equivalently, $\{x : f(x) \leq \alpha\}$ is a closed set, for any $\alpha \in \mathbb{R}$

$$f(x) = \begin{cases} e^{-x}, & \text{if } x < 0 \\ +\infty, & \text{if } x \geq 0 \end{cases}$$



$\text{dom}(f) =]-\infty, 0[, \quad \arg\min_x f(x) = \emptyset$

$$f(x) = \begin{cases} e^{-x}, & \text{if } x \leq 0 \\ +\infty, & \text{if } x > 0 \end{cases}$$



$\text{dom}(f) =]-\infty, 0], \quad \arg\min_x f(x) = \{0\}$

Unless stated otherwise, we only consider l.s.c. functions.

Coercivity, Convexity, and Minima

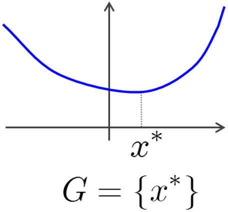
$$f : \mathbb{R}^N \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$$

f is coercive if $\displaystyle \lim_{\|x\| \rightarrow +\infty} f(x) = +\infty$

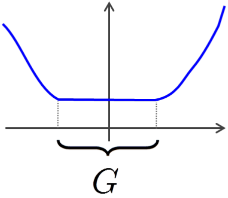
if f is coercive, then $G \equiv \arg\min_x f(x)$ is a non-empty set

if f is strictly convex, then G has at most one element

coercive and strictly convex



coercive, not strictly convex



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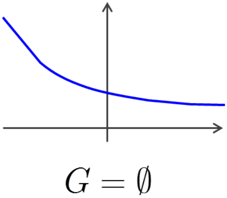
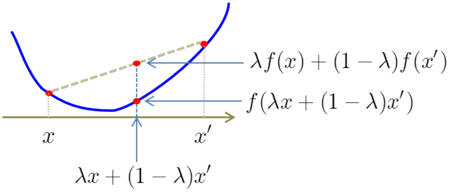


Another Important Concept: Strong Convexity

Recall the definition of convex function: $\forall \lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)x') \le \lambda f(x) + (1-\lambda) f(x')$$



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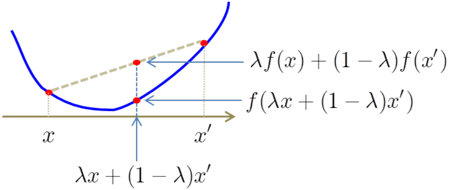

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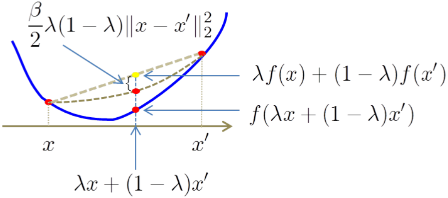

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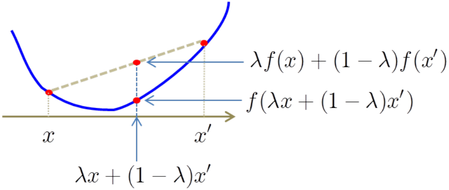

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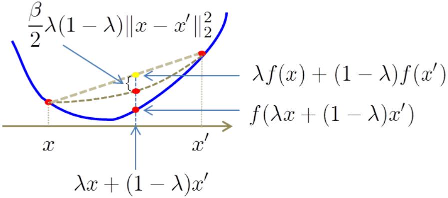

strong convexity

Strong convexity $\underset{\not\Leftarrow}{\Rightarrow}$ strict convexity.

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An important function: the indicator of a set $C \subset \mathbb{R}^n$,

$$\iota_C : \mathbb{R}^n \rightarrow \bar{\mathbb{R}}, \quad \iota_C(x) = \begin{cases} 0 & \Leftarrow \quad x \in C \\ +\infty & \Leftarrow \quad x \notin C \end{cases}$$

If C is a closed convex set, ι_C is a l.s.c. convex function.

The Case of Differentiable Functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be twice differentiable and consider its Hessian matrix at x, denoted $\nabla^2 f(x)$ (or $Hf(x)$):

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- f is strictly convex $\Leftarrow$ its Hessian $\nabla^2 f(x)$ is positive definite $\forall_x$
- f is β-strongly convex $\Leftrightarrow$ its Hessian $\nabla^2 f(x) \succeq \beta I$, with $\beta > 0$, $\forall_x$.

More on the Relationship Between ℓ_1 and ℓ_0

Finding the sparsest solution is NP-hard (Muthukrishnan, 2005).

$$\begin{aligned} \widehat{w} &= \arg\min_{w} \|w\|_0 \\ &\quad \text{s. t. } \|Aw - y\|_2^2 \leq \delta \end{aligned}$$

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Under conditions, replacing ℓ_0 with ℓ_1 yields "similar" results:
see compressive sensing (CS) (Candès et al., 2006; Donoho, 2006)

The Ubiquitous ℓ_1 Norm

- Lasso (*least absolute shrinkage and selection operator*) (Tibshirani, 1996) a.k.a. *basis pursuit denoising* (Chen et al., 1995):

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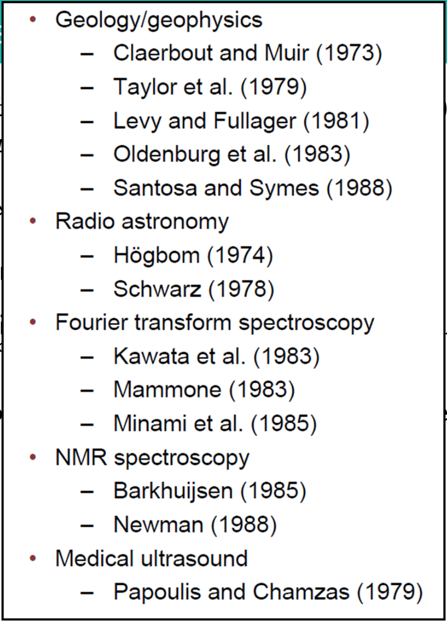

- Geology/geophysics
  - Claerbout and Muir (1973)
  - Taylor et al. (1979)
  - Levy and Fullager (1981)
  - Oldenburg et al. (1983)
  - Santosa and Symes (1988)
- Radio astronomy
  - Högbom (1974)
  - Schwarz (1978)
- Fourier transform spectroscopy
  - Kawata et al. (1983)
  - Mammone (1983)
  - Minami et al. (1985)
- NMR spectroscopy
  - Barkhuijsen (1985)
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from (Goyal et al, 2010)

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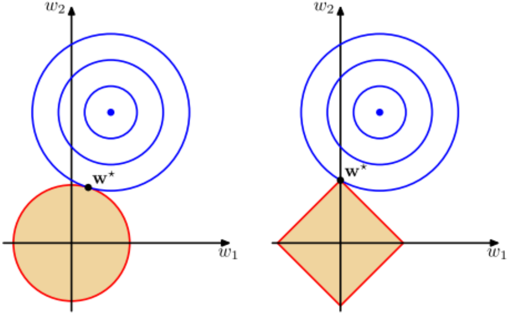
- Why does ℓ_1 yield sparse solutions? (next slides)
- How to solve these problems? (this tutorial)

Why ℓ_1 Yields Sparse Solution

$$\begin{array}{lll} w^* = & \arg\min_w & \|Aw - y\|_2^2 \\ & \text{s.t.} & \|w\|_2 \leq \delta \end{array} \qquad \text{vs} \qquad \begin{array}{lll} w^* = & \arg\min_w & \|Aw - y\|_2^2 \\ & \text{s.t.} & \|w\|_1 \leq \delta \end{array}$$



Why ℓ_1 Yields Sparse Solution

The simplest problem with ℓ_1 regularization

$$\widehat{w} = \arg\min_{w} \frac{1}{2}(w-y)^2 + \lambda|w| = \mathrm{soft}(y,\lambda) = \begin{cases} y-\lambda & \Leftarrow \quad y > \lambda \\ 0 & \Leftarrow \quad |y| \leq \lambda \\ y+\lambda & \Leftarrow \quad y < -\lambda \end{cases}$$

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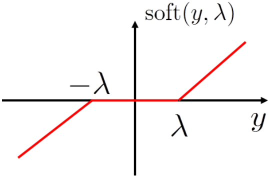

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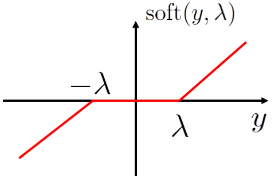

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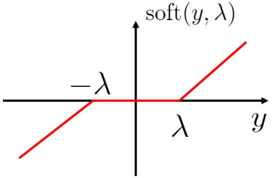

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Contrast with the squared ℓ_2 (ridge) regularizer (linear scaling):

$$\widehat{w} = \arg\min_{w} \frac{1}{2}(w-y)^2 + \frac{\lambda}{2}w^2 = \frac{1}{1+\lambda}\,y$$

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The ℓ_0 “norm” (number of non-zeros): $\|w\|_0 = |\{i : w_i \neq 0\}|$.
Not a norm, not convex, but in the simple case...

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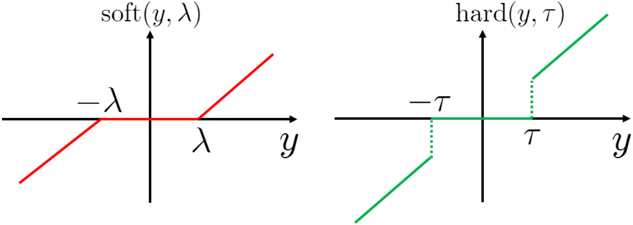

Another Application: Images

Natural images are well represented by a few coefficients in some bases.

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Original 1000×1000 image $x \in \mathbb{R}^{10^6}$

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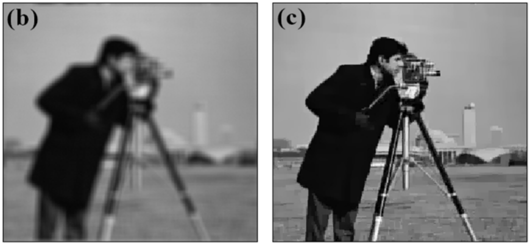
- Also (even more) true with an over-complete tight frame; W is "fat" (more columns than rows) and $WW^T = I$, but $W^T W \neq I$.

Application to Image Deblurring/Deconvolution

blurred

restored



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$$\mathbf{A} = \mathbf{B}\mathbf{W}$$

convolution (blur)

wavelet basis (or tight frame)

Application to Magnetic Resonance Imaging

$$\widehat{\mathbf{x}} \in \arg\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|_2^2 + \tau\,\|\mathbf{x}\|_1$$

$$\mathbf{A} = \mathbf{M}\mathbf{U}\mathbf{W}$$

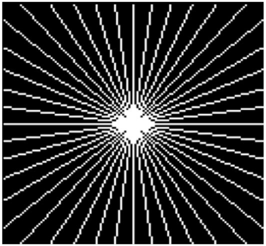
binary mask

discrete Fourier transform

wavelet basis (or tight frame)

original

acquired slices in DFT domain



reconstruction $\mathbf{W}\widehat{\mathbf{x}}$

Machine/Statistical Learning: Linear Regression

Data N pairs $(x_1, y_1), ..., (x_N, y_N)$, where $x_i \in \mathbb{R}^d$ (feature/variable vectors) and $y_i \in \mathbb{R}$ (outputs).

Goal: find "good" linear function: $\widehat{y} = \sum_{j=1}^{d} w_j x_j + w_{d+1} = [x^T 1]w$

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Mean squared error: $\min_w \mathbb{E}\big(Y - [X^T 1] w\big)^2$ impossible! $P_{X,Y}$ unknown

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Regularization: $\min_w \|y - Aw\|_2^2 + \tau\psi(w)$

Machine/Statistical Learning: Linear Classification

Data N pairs $(x_1, y_1), ..., (x_N, y_N)$, where $x_i \in \mathbb{R}^d$ (feature vectors) and $y_i \in \{-1, +1\}$ (labels).

Goal: find "good" linear classifier (*i.e.*, find the optimal weights):

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Convexification: EE neither convex nor differentiable (NP-hard problem).
Solution: replace $h : \mathbb{R} \to \{0, 1\}$ with convex loss $L : \mathbb{R} \to \mathbb{R}_+$.

Machine/Statistical Learning: Linear Classification

Criterion: $$\underbrace{\min_{w} \sum_{i=1}^{N} L\big(\underbrace{y_i\,(w^T x_i + b)}_{\text{margin}}\big)}_{f(w)} + \tau\psi(w)$$

Regularizer: $\psi = \ell_1 \quad \Rightarrow$ encourage sparseness $\quad \Rightarrow$ feature selection

Convex losses: $L : \mathbb{R} \to \mathbb{R}_+$ is a (preferably convex) loss function.

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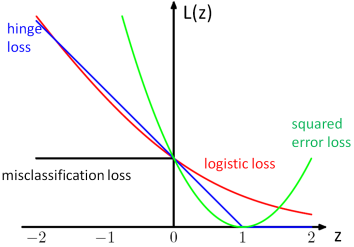
Convex losses: $L : \mathbb{R} \to \mathbb{R}_+$ is a (preferably convex) loss function.

- Misclassification loss: $L(z) = 1_{z<0}$
- Hinge loss: $L(z) = \max\{1 - z, 0\}$
- Logistic loss: $L(z) = \frac{\log\left(1+\exp(-z)\right)}{\log 2}$
- Squared loss: $L(z) = (z-1)^2$



Machine/Statistical Learning: General Formulation

This formulation covers a wide range of linear ML methods:

$$\min_{w} \underbrace{\sum_{i=1}^{N} L\big(y_i\ ([x^T\ 1]w)\big)}_{f(w)} + \tau\psi(w)$$

- Least squares regression: $L(z) = (z-1)^2,\ \ \psi(w) = 0.$
- Ridge regression: $L(z) = (z-1)^2,\ \ \psi(w) = \|w\|_2^2.$
- Lasso regression: $L(z) = (z-1)^2,\ \ \psi(w) = \|w\|_1$
- Logistic regression: $L(z) = \log(1+\exp(-z))$ (ridge, if $\psi(w) = \|w\|_2^2$
- Sparse logistic regression: $L(z) = \log(1+\exp(-z)),\ \ \psi(w) = \|w\|_1$
- Support vector machines: $L(z) = \max\{1-z, 0\},\ \ \psi(w) = \|w\|_2^2$
- Boosting: $L(z) = \exp(-z)$,
- ...

Machine/Statistical Learning: Nonlinear Problems

What about non-linear functions?

Simply use $\widehat{y} = \phi(x, w) = \sum_{j=1}^{D} w_j \, \phi_j(x)$, where $\phi_j : \mathbb{R}^d \to \mathbb{R}$

Essentially, nothing changes; computationally, a lot may change!

$$\min_w \underbrace{\sum_{i=1}^{N} L\big(y_i \; \phi(x, w)\big)}_{f(w)} + \tau \psi(w)$$

Key feature: $\phi(x, w)$ is still linear with respect to w, thus f inherits the convexity of L.

Examples: polynomials, radial basis functions, wavelets, splines, kernels,...

Recover the linear case, letting $D = d + 1$, $f_j(x) = x_j$, and $f_{d+1} = 1$.

Structured Sparsity

ℓ_1 regularization promotes sparsity

A very simple sparsity pattern: prefer models with small cardinality

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Group/structured regularization.

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Main goal: to promote structural patterns, not just penalize cardinality

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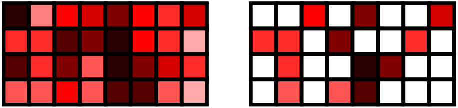
Many applications:

- feature template selection (Martins et al., 2011)
- multi-task learning (Caruana, 1997; Obozinski et al., 2010)
- learning the structure of graphical models (Schmidt and Murphy, 2010)

“Grid” Sparsity

For feature spaces that can be arranged as a grid (examples next)



dense sparse

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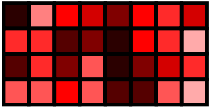

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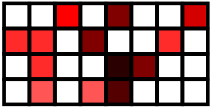

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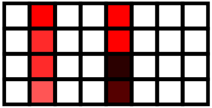

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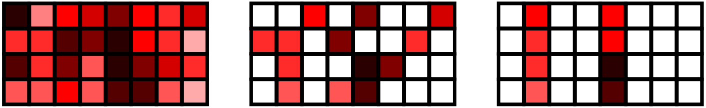

dense

sparse

group sparse

Goal: push *entire columns* to have zero weights

The groups are the columns of the grid

Example: Sparsity with Multiple Classes

In multi-class (more than just 2 classes) classification, a common formulation is

$$\widehat{y} = \arg \max_{y \in \{1,...,K\}} x^T w_y$$

Weight vector $w = (w_1, ..., w_K) \in \mathbb{R}^{Kd}$ has a natural group/grid organization:

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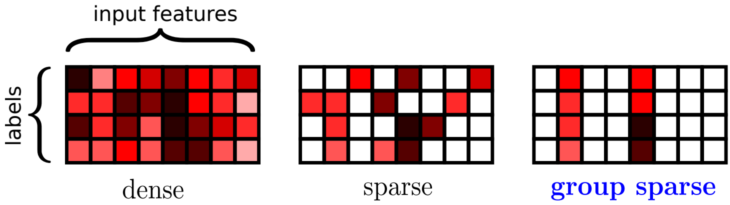


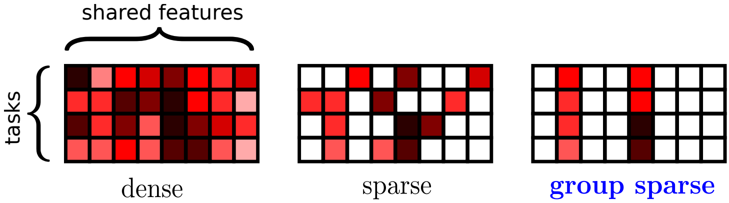
Simple sparsity is wasteful: may still need to keep all the features

Structured sparsity: discard some input features (feature selection)

Example: Multi-Task Learning

Same thing, except now rows are tasks and columns are **features**

Example: simultaneous regression (seek function into $\mathbb{R}^d \to \mathbb{R}^b$)



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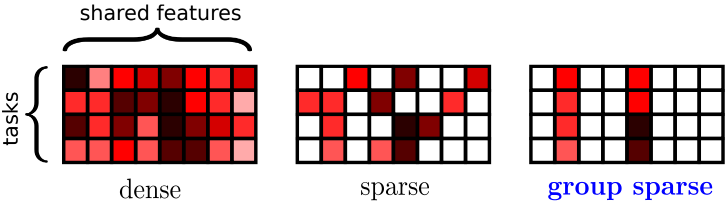
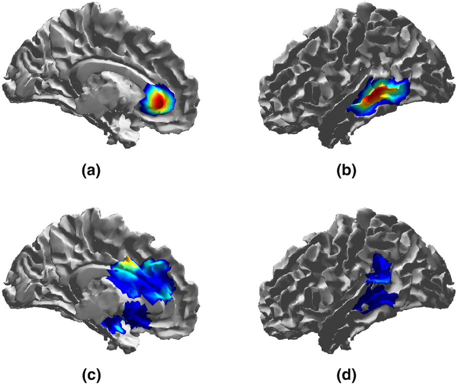


Goal: discard features that are irrelevant for *all* tasks

Approach: one group per feature (Caruana, 1997; Obozinski et al., 2010)

Example: Magnetoencephalograpy (MEG)

Group: localized cortex area at localized time period (Bolstad et al., 2009)



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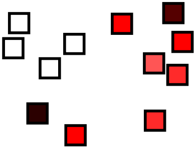

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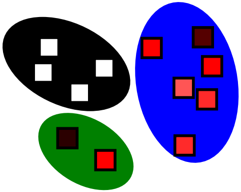

- D features
- M groups $G_1, \ldots, G_M$, each $G_m \subseteq \{1, \ldots, D\}$
- parameter subvectors $x_{G_1}, \ldots, x_{G_M}$

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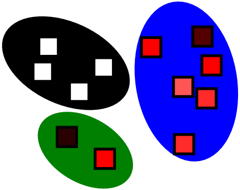

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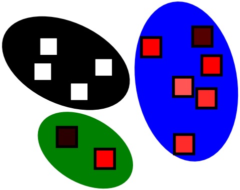

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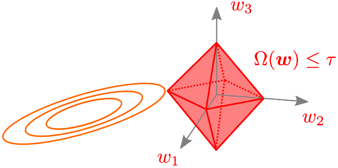
- Intuitively: the ℓ_1 **norm** of the ℓ_2 **norms**
- Technically, still a norm (called a *mixed* norm, denoted $\ell_{2,1}$)

Lasso versus group-Lasso



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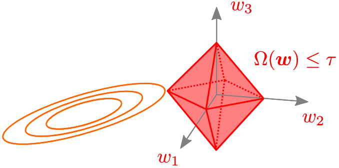

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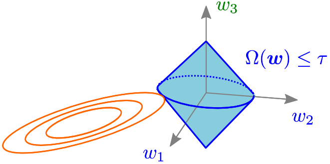
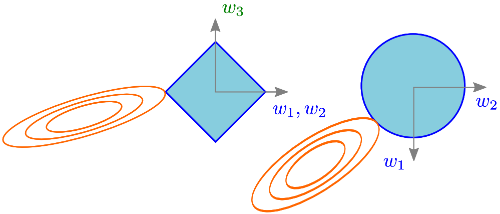


$$\boxed{\Omega(\boldsymbol{w}) = \sqrt{w_1^2 + w_2^2} + |w_3|}$$



Composite Absolute Penalties (Zhao et al., 2009)

A mixed-norm regularization:

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- The most common choice: $\ell_{2,1}$ norm
- Another frequent choice: $\ell_{\infty,1}$ norm (Quattoni et al., 2009; Graça et al., 2009; Eisenstein et al., 2011; Wright et al., 2009)

Three Scenarios

- Non-overlapping Groups
- Tree-structured Groups
- Graph-structured Groups

Non-overlapping Groups

Assume that $G_1, \ldots, G_M$ (where $G_m \subset \{1, ..., d\}$) constitute a partition:

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Examples of non-trivial groups:

- label-based groups
- task-based groups

Tree-Structured Groups

Assumption: if two groups overlap, one is contained in the other
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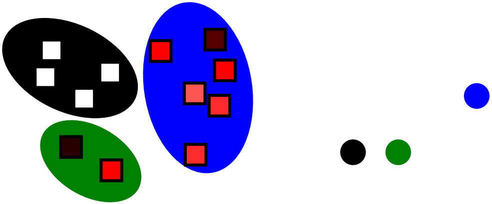

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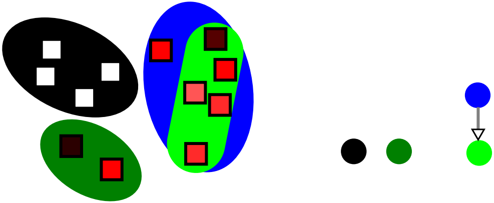

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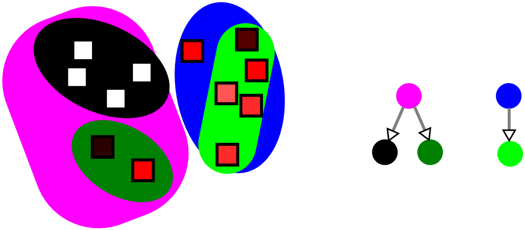

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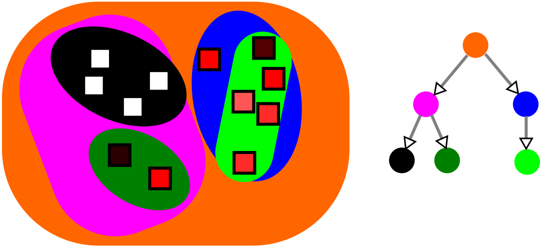

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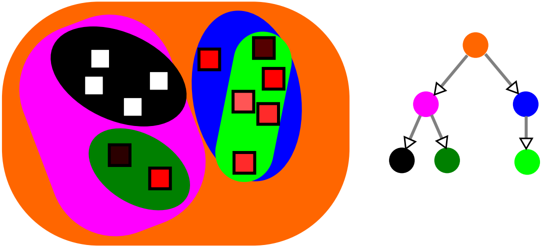

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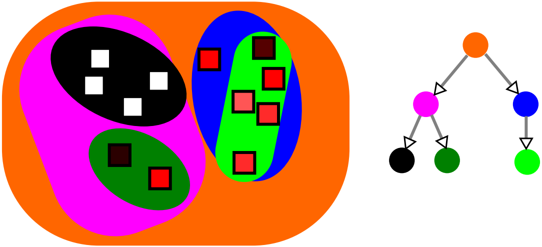

- What is the **sparsity pattern**?
- If a group is discarded, all its descendants are also discarded

Matrix Inference Problems

Sparsest solution:

- From $Bx = b \in \mathbb{R}^p$, find $x \in \mathbb{R}^n$ $(p < n)$.
- $\min_x \|x\|_0 \text{ s.t. } Bx = b$
- Yields exact solution, under some conditions.

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Both $NP-$hard (in general); the same is true of noisy versions:

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Under some conditions, the same solution is obtained by replacing rank(X) by the nuclear norm $\|X\|_*$ (as any norm, it is convex) (Recht et al., 2010)

Matrix Nuclear Norm (and Other Norms)

- Also known as trace norm; the ℓ_1-type norm for matrices $X \in \mathbb{R}^{m \times n}$
- Definition: $\|X\|_* = \text{trace}\left(\sqrt{X^T X}\right) = \sum_{i=1}^{\min\{m,n\}} \sigma_i,$

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- Particular case of Schatten q-norm: $\|X\|_q = \left(\displaystyle\sum_{i=1}^{\min\{m,n\}} (\sigma_i)^q\right)^{1/q}.$

Matrix Nuclear Norm (and Other Norms)

- Also known as trace norm; the ℓ_1-type norm for matrices $X \in \mathbb{R}^{m \times n}$
- Definition: $\|X\|_* = \text{trace}\big(\sqrt{X^T X}\big) = \sum_{i=1}^{\min\{m,n\}} \sigma_i,$

  the σ_i are the singular values of X.
- Particular case of Schatten q-norm: $\|X\|_q = \left(\sum_{i=1}^{\min\{m,n\}} (\sigma_i)^q \right)^{1/q}.$
- Two other notable Schatten norms:
  - Frobenius norm: $\|X\|_2 = \|X\|_F = \sqrt{\sum_{i=1}^{\min\{m,n\}} (\sigma_i)^2} = \sqrt{\sum_{i,i} X_{i,j}^2}$
  - Spectral norm: $\|X\|_\infty = \max\left\{\sigma_1, ..., \sigma_{\min\{m,n\}}\right\}$

Nuclear Norm Regularization

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Linear observations: $\mathcal{B} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$, $\big(\mathcal{B}(X)\big)_i = \langle B_{(i)}, X\rangle$,

$$B_{(i)} \in \mathbb{R}^{m\times n}, \text{ and } \langle B, X\rangle = \sum_{ij} B_{ij}X_{ij} = \text{trace}(B^T X)$$

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Why does the nuclear norm favor low rank solutions? Let $Y = U\Lambda V^T$ be the singular value decomposition, where $\Lambda = \text{diag}\big(\sigma_1, ..., \sigma_{\min\{m,n\}}\big)$; then

$$\arg\min_X \frac{1}{2}\|Y - X\|_F^2 + \tau\|X\|_* = U \underbrace{\text{soft}(X, \tau)}_{\text{may yield zeros}} V^T$$

...singular value thresholding (Ma et al., 2011; Cai et al., 2010)

Another Matrix Inference Problem: Inverse Covariance

Consider n samples $y_1, ..., y_n \in \mathbb{R}^d$ of a Gaussian r.v. $Y \sim \mathcal{N}(\mu, C)$; the log-likelihood is

$$L(P) = \log p(y_1, ..., y_n | P) = \log\det(P) - \text{trace}(SP) + \text{constant}$$

where $S = \frac{1}{n}\sum_{i=1}^{n}(y_i - \mu)(y_i - \mu)^T$ and $P = C^{-1}$ (inverse covariance).

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Zeros in P reveal conditional independencies between components of Y:

$$P_{ij} = 0 \quad \Leftrightarrow \quad Y_i \perp\!\!\!\perp Y_j | \{Y_k,\ k \neq i, j\}$$

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Sparsity (presence of zeros) in P is encouraged by solving

$$\min_{P \succ 0} \underbrace{-\log\det(P) + \text{trace}(SP)}_{f(P)} + \tau \underbrace{\|\text{vect}(P)\|_1}_{\psi(P)}$$

where $\text{vect}(P) = [P_{1,1}, ..., P_{d,d}]^T$.

Atomic-Norm Regularization

Key concept in sparse modeling: synthesize "object" using a few atoms:

$$x = \sum_{i=1}^{|\mathcal{A}|} c_i \, a_i$$

- $\mathcal{A}$ is the set of atoms (the atomic set), or building blocks.
- $c_i \geq 0$ are weights; x is simple/sparse object $\Rightarrow \quad \|c\|_0 \ll |\mathcal{A}|$
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The (Minkowski) gauge of $\mathcal{A}$ is:

$$\|x\|_{\mathcal{A}} = \inf\big\{ t > 0 : \; x \in t \, \mathrm{conv}(\mathcal{A}) \big\}$$

Assuming that $\mathcal{A}$ centrally symmetry about the origin ($a \in \mathcal{A} \;\Rightarrow\; -a \in \mathcal{A}$), $\| \cdot \|_{\mathcal{A}}$ is a norm, called the atomic norm Chandrasekaran et al. (2012).

Atomic-Norm Regularization

The atomic norm

$$\|x\|_{\mathcal{A}} = \inf\big\{ t > 0 : \; x \in t\,\mathrm{conv}(\mathcal{A}) \big\}$$
$$= \inf\Big\{ \sum_{i=1}^{|\mathcal{A}|} c_i : \; x = \sum_{i=1}^{|\mathcal{A}|} c_i\, a_i, \; c_i \geq 0 \Big\}$$

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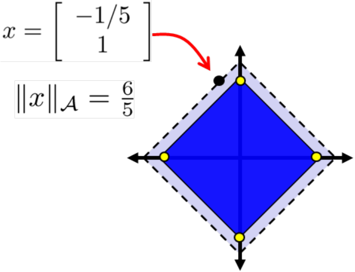
Example: the ℓ_1 norm as an atomic norm

- $\mathcal{A} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right\}$
- $\text{conv}(\mathcal{A}) = B_1(1)$ (ℓ_1 unit ball).
- $\begin{aligned}\|x\|_{\mathcal{A}} &= \inf\big\{t > 0 : \; x \in t\, B_1(1)\big\} \\ &= \|x\|_1\end{aligned}$



$x = \begin{bmatrix} -1/5 \\ 1 \end{bmatrix}$

$\|x\|_{\mathcal{A}} = \frac{6}{5}$

Atomic Norms: More Examples

Examples with easy forms:

- *sparse vectors*

  $$\mathcal{A} = \{\pm e_i\}_{i=1}^N$$

  $$\mathrm{conv}(\mathcal{A}) = \text{cross-polytope}$$

  $$\|x\|_{\mathcal{A}} = \|x\|_1$$

- *low-rank matrices*

  $$\mathcal{A} = \{A : \mathrm{rank}(A) = 1, \|A\|_F = 1\}$$

  $$\mathrm{conv}(\mathcal{A}) = \text{nuclear norm ball}$$

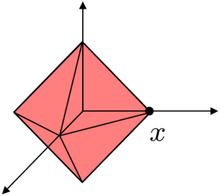
  $$\|x\|_{\mathcal{A}} = \|x\|_\star$$

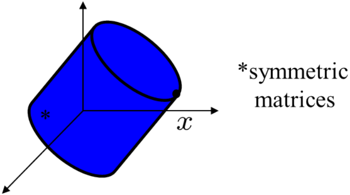
- *binary vectors*

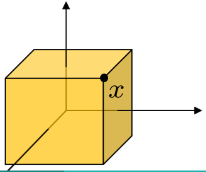
  $$\mathcal{A} = \{\pm 1\}^N$$

  $$\mathrm{conv}(\mathcal{A}) = \text{hypercube}$$

  $$\|x\|_{\mathcal{A}} = \|x\|_\infty$$







Atomic-Norm Regularization

Given an atomic set $\mathcal{A}$, we can adopt an Ivanov formulation

$$\min f(x) \text{ s.t. } \|x\|_{\mathcal{A}} \leq \delta$$

(for some $\delta > 0$) tends to recover x with sparse atomic representation.

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Yes! The conditional gradient (more later.)

Summary

- Many inference, learning, signal/image processing problems can be formulated as optimization problems.
- Sparsity-inducing regularizers play an important role in these problems
- There are several way to induce sparsity
- It is possible to formulate structured sparsity
- It is possible to extend the sparsity rationale to other objects, namely matrices
- Atomic norms provide a unified framework for sparsity/simplicity regularization

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